Learning Continuum for Place-Value

From *Cognition-Based Assessment and Teaching of Place-Value* by Michael Battista, 2012.

- CBA materials are designed to help students move to higher levels of reasoning. It is important, however that instruction not *demand* that students 'move up" the levels of sophistication with insufficient cognitive support.
- "Jump in levels" are made internally by students, not by teachers or the curriculum. This does not mean that students must progress with no help. Teaching helps students by providing them with the right kinds of encouragement, support and challenges.
- Good teaching has students work on problems that stretch, but do not overwhelm, their reasoning; asks good questions, has students discuss their ideas with other students; and sometimes shows students ideas that they don't invent themselves.
- Before students attempt to understand place value, they must first understand whole numbers as telling how many objects are in a set. And they must understand counting as setting up a one-to-one correspondence between a sequence of counting words and the objects in the set.
- To understand place value students must understand place value in individual numbers and understand place value in computational algorithms.
- Counting is a foundation for the development of students' understanding of number. Students begin by counting by ones to determine the number of objects in a set or group. Students begin by counting, progress to skip-counting, skip –counting by place value parts and then replace counting with procedures for combining and separating numbers.
- Counting fingers is more sophisticated than counting objects and counting "counting words" is more abstract than counting fingers. For this reason we should also note if a student needs physical or visible materials to complete a task.
- Students develop conceptual understanding of computational algorithms when their understanding of place value and properties of numbers become sophisticated enough to guide and make sense of the sequence of steps in algorithms.
- The major reason traditional computational algorithms are difficult for students to make sense of – and the primary cause for many procedural errors with these algorithms – is that place-value ideas are often "hidden".

Levels of Sophistication in Student Reasoning: Place Value

- These levels describe how students progress from a beginning understanding of addition and subtraction concepts to meaningful use of addition and subtraction algorithms.
- The "jumps" between sublevels are small enough that students can achieve them with small amounts of instruction in relatively short periods of time. Sublevels serve as accessible stepping-stones in students' development.
- CBA levels focus on the development of concepts and reasoning in place value. The CBA levels develop gradually as you study examples of student's work and as you use CBA with your students.

Place Value	
Level 0	Student has difficulties counting by ones Task: How many cubes are there? Response: Student either counts cubes twice or fails to count all cubes.
Level 1	Student operates on numbers as collections of ones (no skip-counting by place value)
1.1	Student correctly counts groups of objects by ones but cannot count groups of ten. Task: How many cubes are there? Response: 1,2,3,4,… 20,21,22,23. Student correctly counts based on one-to-one correspondence.
1.2	Student correctly counts groups of tens. They treat groups of ten ones as countable units, however, they continue to count by ones not tens. They are able to count groups of ten when using visible materials. Task: There are 38 cookies. How many bags of 10 cookies can be made? Response: Student uses place value blocks and individually counts by ones using the ten blocks. 1,2,3 … 10,11,12 … 20, 21, 22 …30. Counts individual one blocks 31,32…38. Counts the ten blocks to get the answer 3.
1.3	Student operates on tens and ones separately as ones. (They do not understand that 1 ten equal to ten ones. Algorithms are learned by rote and are not understood and not always correctly carried out.) Task: 39 + 46 = Response: 3 + 4 = 7 (writes 7). 9 + 6 = 15, and I don’t know if I should write the 1 or the 5. So I am going with the 5 because my mom told me to use the second number (writes 5). So I say 75.
Level 2	Student operates on numbers by skip-counting by place value. (e.g. counts by tens)
2.1	Student counts by tens and ones separately Task: Given 4 ten blocks and 7 one blocks, how many squares are there altogether? Response: 10,20,30,40,41,42 … 47 [Teacher adds two more 10 blocks] Response: 10,20,30,40,50,60,61,62 … 67 (Student does not count on from 37)

2.2	Student counts by tens in mid-decades Task: In a box, there are 35 red apples and 27 green apples. How many apples are in the box? Response: Use place value blocks to model. 35,45,55, (pointing to ones) 60, 62.
Level 3	Student operates on numbers by combining and separating place value parts (e.g. adds tens parts without counting)
3.1	Student uses multiples-of-ten language (forty plus twenty equals sixty) Task: In a box, there are 35 red apples and 27 green apples. How many apples are in the box? Response: 30 plus 20 is 50, 5 and 7 is 12. So 50 and 12 is 62.
3.2	Student uses tens language (4 tens plus 2 tens equals 6 tens) Task: A number has 14 ones and 3 tens. What is the number? Response: 3 tens plus another ten from 14 is 4 tens, plus the 4 ones is 44.
3.3	Student integrates levels 2.1 – 3.2 Task: a number has 23 tens and 6 ones. What is the number? Response: twenty tens is 2 hundred and 3 tens. So that's two hundred plus thirty, which is 230. And 6 ones, that's 236.
Level 4	Student understands place value in expanded algorithms
4.1	Student understands place value in expanded addition and subtraction algorithms. $\mathbf{300 + 40 + 2}$ $\mathbf{+\ 400 + 30 + 5}$ Task: 342 + 435 $\mathbf{700 + 70 + 7 = 777}$ Response:
4.2	Student understands place value in expanded multiplication and division algorithms. $\mathbf{30 \times 40 = 1200}$ Task: 45 × 34 $\mathbf{30 \times 5 = 150}$ $\mathbf{4 \times 40 = 160}$ Response: $\mathbf{4 \times 5 = 20}$ $\mathbf{= 1530}$

Level 5	Student understand place value in traditional algorithms
5.1	Student understands place value in traditional addition and subtraction algorithms. Task: 63 – 38 Response: 3 – 8 you cant do that, it'd be negative. So I'm gonna make this a 50 and put the ten in the 3, so that's 13. Now 13 – 8 = 5. And 50 – 30 = 20. So that's 25.
5.2	Student understands place value in traditional multiplication and division algorithms. Task: 45 × 23 Response: 3 times 5 is 15 (writes 5), put the ten up here in the tens column. 3 times 4 is 12, plus 1 is 13 (writes 13). 2 times 5 is 10, write the 0 (puts it in the tens column0, put the one above the 4. 2 times 4 is 8 plus 1 is 9. adds the numbers to get 1035.
Level 6	Student generalizes place-value understanding to larger numbers, numbers less than 1, and exponential notation. Students can properly name large numbers taking into account the decimal point.

Develop a Profile of a Student's Reasoning about Place Value

- CBA assessment tasks are designed to help you assess levels of reasoning, not levels of students.
- A student might use one level of reasoning of visually presented tasks, and another on symbolic tasks.
- CBA Reasoning Profile
    1. Record what the student did on CBA problems
    2. Construct a CBA levels summary chart for student.
    3. Develop a set of goals and recommendations for instruction.

CBA Assessment Tasks for Addition and Subtraction

- The problem sheets can be used in individual interviews with children or in class as instructional activities. However, no matter what you choose, it is critical to get the students to write and describe or discuss their strategies. Only then can you use the CBA levels to interpret students' responses and decide on needed instruction.
- The purpose of interviewing students with CBA tasks is to determine how they are reasoning and more specifically, to determine what CBA levels of reasoning they are using for the tasks.

Instructional Strategies for Place Value

- For students to make progress, have them do several problems of a specific type until you see them move to the next level, or you become convinced that they are not quite ready to move on to the next level.
- Attaining higher levels of reasoning about place value, especially the highest levels, is interrelated with attaining higher levels of reasoning about the arithmetic operations of addition, subtraction, multiplication and division.

Teaching Students at Level 0: Constructing Initial Meaning for Object Counting and Numbers.

- Help students learn the verbal count-by-ones sequence.
- Students have particular difficulty learning the names of numbers in the teens, and at transitions between multiples of ten and at transitions between hundreds.
- One way you can help students learn the verbal sequence is by doing class counts: "Let's all count from 1 to 20". Do class counts as you point to a set of objects.
- Students must learn to organize their counting so that each object in a set is counted once and only once.
- Once students can reliably count sets of objects, give them problems to help them understand the relationship between counting and the numerosity (cardinality) of sets: "When we count a set of objects, the last number recited tells how many objects are in the set".

Teaching Students at Level 1: Learning to Count Groups of Ten and Skip-Count by Ten.

Level 1.1: Moving to Counting Groups of Ten

- Use physical materials such as unifix or multilink cubes to pose problems in which students count groups of ten. Don't use place-value blocks yet – it's important for students to create the group of ten.

Level 1.2 or 1.3: Moving to Skip-Counting by Tens.

- To progress from counting by ones to counting by tens, students must come to see the counting-by-tens words – *ten, twenty, thirty* – as signifying the number ones in successive multiples of ten.
- Activities to build understanding:
    - Using ten-strips
    - Use place-value blocks
    - Use hundred chart
    - Use place-value house problems.

Teaching Students at Level 2: Extending Understanding of Skip-Counting by Ten

Level 2.1: Moving to Counting by Tens and Ones in Mid-Decades

- Students must extend their knowledge of counting patterns from counting by tens and ones separately to counting by tens and ones in mid-decades.
- Activities to build understanding include:
    - Hidden ten strips
    - Add 2 two-digit numbers that are not multiples of ten
    - Increase or decrease various numbers by multiples of ten
    - Tasks with number houses.
- When students are successful with visual models such as the hundred chart and house problems, encourage them to extend their counting-by-tens reasoning to symbolically represented addition and subtraction problems.

Level 2.2: Moving to Combining and Separating Place-Value Parts (No Skip Counting)

- Problems to encourage proper progression
    - Deriving tens problems form basic facts
    - Doing problems with place-value blocks
    - Do problems with ten strips without counting
    - Do problems mentally, without counting.

Teaching Students at Level 3: Increasing Understanding of Combining and Separating by Place Value.

Level 3.1: Moving to Developing Tens Language

- Encourage students to use ones, tens and hundreds language as required for Level 3.2.
- Students who are reasoning at Level 3.1 will make sense of this language by converting it into multiples of ten language
- You can encourage student who are combining and separating numbers using multiples of ten language to move to tens language by exposing them to tens language.
- If necessary, students can use place-value blocks to solve these problems. But the goal is for students to be able to do the problems without physical materials or pictures.

Level 3.2: Moving to Developing Integrated Language and Strategy Use

- Have students discuss the different strategies that can be used to add and subtract two-digit numbers.
- If you conduct enough class discussion in which students explain and reflect on the different strategies, students will come to accept all as valid, and come to see them as equivalent.

Level 3.3: Moving to Understanding Place Value in Expanded Algorithms

- Before students can genuinely understand algorithms, they must be able to mentally combine and separate numbers by their place-value parts as in Level 3,1 through Level 3.3.
- Those who use place-value blocks are not ready to learn algorithms because the student does not regroup, this work with place-value blocks cannot serve as an appropriate mental model or justification for a traditional addition algorithm that requires regrouping.
- To encourage regrouping when using place-value blocks, you have to add an extra rule for using blocks – whenever there are ten of any one kind of block, you must regroup.
- Place-value blocks cannot serve as an appropriate model for traditional subtraction algorithm.
- Once students are able to reason at Level 3.3 without needing physical or visual material, they should be encouraged to move to expanded, conceptually explicit algorithms for addition and subtraction.

Teaching Students at Level 4: Moving From Expanded to Traditional Algorithms

- There are several ways that you can help students make sense of traditional algorithms conceptually. One way is to model the algorithm with place-value blocks. Doing steps in the base-ten block procedure at the same time as corresponding steps in the algorithm encourages students to develop conceptual meaning for the algorithms.
- Another way is to replace the abbreviated language used in implementing traditional algorithms with conceptually explicit language.
- The final way, and most meaningful way to most students is to have students relate the traditional algorithms step by step to expanded algorithms with explicit place-value language.

Level 4.1: Moving to Understanding Traditional Algorithms for Addition and Subtraction.

- With both traditional and expanded algorithms visible to students, ask questions to encourage students to see precisely how the steps in the traditional algorithm correspond to steps in the expanded algorithm.
- A similar connection can be made between the traditional subtraction algorithm and an expanded subtraction algorithm.

Level 4.2: Moving to Understanding Traditional Algorithms for Multiplication

- The language the student uses in performing the algorithm does not make the place values of digits evident. Place value is dealt by the positioning of digits in partial products.
- The traditional algorithm reduces the problem of multiplying multidigit numbers to operating on single digits by using basic multiplication facts and appropriate spatial positioning of the digits in the result.

Level 4.3: Moving to Understanding Traditional Algorithms for Division

Teaching Students at Level 5: Moving to Understanding Larger Numbers, Numbers Less than 1, and Exponential Notation.

- Students deepen and extend their understanding of number naming schemes, of relationships between place values and of the exponential format for expressing numbers as a sum of their place-value parts.